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# A Remark on Schrödinger Operators on Aperiodic Tilings

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We prove that for a large class of Schrödinger operators on aperiodic tilings the spectrum and the integrated density of states are the same for all tilings in the local isomorphism class, i.e., for all tilings in the orbit closure of one of the tilings. This generalizes the argument in earlier work from discrete strictly ergodic operators on $l^2(\mathbb{Z}^d)$ to operators on the $l^2$-spaces of sets of vertices of strictly ergodic tilings.

**KEY WORDS:** Discrete Schrödinger operators; integrated density of states, spectrum, Penrose tilings, projection method, unique ergodicity; minimality.

## 1. INTRODUCTION

For any Penrose tiling $\tau$ one can consider a discrete Schrödinger operator $H_\tau$ on $l^2(V_\tau)$, the $l^2$-space of the set of vertices $V_\tau$ of $\tau$ defined by

$$(H_\tau \psi)(n) = \sum_{\langle m,n \rangle} \psi(m) + V(n)\,\psi(n) \tag{1.1}$$

where the summation extends over the nearest neighbors of $n$. The potential $V(n)$ is real and depends on the type of the vertex $n$, or, more generally, on the neighborhood of the vertex up to some radius $r$ (independent of $n$). Operators of this form have been studied as possible models of electronic properties of quasicrystals (for references, see ref. 1). In a previous paper,[1] we proved that the integrated density of states of $H_\tau$ exists for every Penrose tiling $\tau$ and is independent of $\tau$. The proof in ref. 1 uses the "self-similarity" of Penrose tiling and applies to Schrödinger operators on every

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(local isomorphism) class of tilings that are self-similar in the way Penrose tilings are self-similar.

It is well known (see, e.g., ref. 2) that for ergodic discrete Schrödinger operators on $l^2(\mathbb{Z}^d)$ the spectrum and the integrated density of states are almost surely independent of the realization of the potential. In ref. 1 we proved they are actually independent (not just almost surely) of the potential for strictly ergodic operators (i.e., operators for which the closure of the set of all translates of the potential is uniquely ergodic and minimal with respect to translations).

This note generalizes the latter result to Schrödinger operators on strictly ergodic tilings. For self-similar tilings this gives a new proof that the integrated density of states is independent of the tiling. The new proof is more general: it also applies to Schrödinger operators on tilings defined by the projection method (see, e.g., refs. 3 and 4). Moreover, for both self-similar tilings and tilings defined by the projection method we get the new result that the spectrum is independent of the tiling.

Schrödinger operators on $l^2(\mathbb{Z}^d)$ all act on the same Hilbert space. But operators $H_\tau$ and $H_{\tau'}$ act on different Hilbert spaces—unless $\tau$ and $\tau'$ differ only by a translation; moreover, there is no unitary action of translations on the Hilbert space. This is the reason why the proofs given in ref. 1 for strictly ergodic operators on $l^2(\mathbb{Z}^d)$ need modification before they apply to operators on the $l^2$ spaces of sets of vertices of aperiodic strictly ergodic tilings.

Before stating and proving the result we describe which tilings we are dealing with.

## 2. STRICTLY ERGODIC TILINGS

A tiling of $\mathbb{R}^d$ is a covering of $\mathbb{R}^d$ by closed sets (tiles) such that the interiors of the tiles are pairwise disjoint. We assume (mainly for simplicity) that the tiles are convex polytopes matching face to face and that modulo translations there are finitely many different kinds of tiles.

Let $\tau$ be such a tiling. A patch of $\tau$ is a finite set of tiles occurring in $\tau$. Denote by $C_L$ the cube of side $L$ centered at 0. For $\Lambda \subset \mathbb{R}^d$ let $N_P(\Lambda)$ be the number of copies of the patch $P$ inside $\Lambda$. The tiling $\tau$ is called strictly ergodic if for every patch $P$ the frequency $n_P := \lim_{L\to\infty} L^{-d} N_L(C_L + a)$ exists uniformly in $a \in \mathbb{R}^d$ and $n_P > 0$. Self-similar tilings and tilings generated by the projection method are strictly ergodic in this sense, as was shown in refs. 5 and 4, respectively.

For $x \in \mathbb{R}^d$ denote the translated tiling $\tau + x$ by $T_x \tau$ (here and below a tiling and its translates are considered different tilings). Close the set $\{T_x \tau \mid x \in \mathbb{R}^d\}$ in the metric $d(\tau, \tau') = \min(1, \varepsilon)$, where $\varepsilon$ is the smallest number such that there is a vector $x$ of Euclidian norm $\|x\| \leqslant \varepsilon$ such that $\tau$ and $T_x \tau'$ coincide on the sphere of radius $1/\varepsilon$ around 0. This gives a shift-invariant set $\Omega_\tau$ that is compact as a closed subset of a complete metric space.[6,7]

The set $\Omega_\tau$ is strictly ergodic with respect to translation in the sense "strictly ergodic" is used in ergodic theory,[8] i.e., uniquely ergodic and minimal. In particular, we have that for all continuous functions on $\Omega_\tau$

$$\lim_{L \to \infty} L^{-d} \int_{C_L} f(T_x \sigma)\, dx = \int f\, d\mu \qquad \text{uniformly in} \quad \sigma \in \Omega_\tau \tag{2.1}$$

(unique ergodicity) and that $\mu(A) > 0$ for every open $A \subset \Omega$ (minimality for uniquely ergodic systems). These two statements are the analogs of Propositions 7.1 and 7.2 in ref. 1, respectively.

## 3. RESULT

Let $\Omega$ be a strictly ergodic tiling dynamical system. Denote the set of vertices of $\tau \in \Omega$ by $V_\tau$. A finite subset of $V_\tau$ will be called a vertexpattern. For $r \geqslant 0$, the $r$-environment[1] $E_r(A)$ of a vertexpattern $A$ is defined as $E_r(A) := \{x \in V_\tau \mid \|x - y\| \leqslant r \text{ for some } y \in V_\tau\}$.

Let $H_\tau$ be as in (1.1). Denote its resolution of the identity by $E_\tau(\cdot)$. Let $\chi_L \in l^2(V_\tau)$ denote the characteristic function of the set of vertices lying within the cube $C_L$; so $\chi_L$ implicitly depends on $\tau$. For each $\tau \in \Omega$ define measures $k_L^\tau$ on $\mathbb{R}$ by $k_L^\tau(A) := L^{-d} \operatorname{tr}(E_\tau(A)\, \chi_L)$. There is an $\varepsilon > 0$ such that $\|x - y\| > 2\varepsilon$ for all $x, y \in V_\tau$ and all $\tau \in \Omega$. Let $B_\varepsilon := \{x \in \mathbb{R}^d \mid \|x\| \leqslant \varepsilon\}$. Let $\psi$ be a continuous, nonnegative real function on $\mathbb{R}^d$ with support in $B_\varepsilon$ and $\int \psi\, dx = 1$. For continuous real functions $f$ of compact support on $\mathbb{R}$ define $\bar{f}$ on $\Omega$ by

$$\bar{f}(\tau) := \begin{cases} \psi(v)(\delta_v, f(H_\tau)\, \delta_v) & \text{if there is a } v \in V_\tau \text{ such that } v \in B_\varepsilon \\ 0 & \text{otherwise} \end{cases} \tag{3.1}$$

The integrated density of states is the distribution function of the measure $k$ defined by $\int f\, dk := \int \bar{f}\, d\mu$.

**Theorem.** Let $\Omega$ be a strictly ergodic tiling dynamical system and let $H_\tau$ be as in (1.1). Then for all $\tau \in \Omega$:

(i) $k_L^\tau \to k$ vaguely as $L \to \infty$.

(ii) The spectrum of $H_\tau$ is equal to the topological support of $k$.

*Proof.* We prove (i); the proof of (ii) is then analogous the proof of Proposition 7.4 in ref. 1.

We have to show that for every real continuous function $f$ on $\mathbb{R}$ of compact support $\lim_{L\to\infty} \int f\,dk_L^\tau = \int \bar{f}\,d\mu$. Now

$$\int f\,dk_L^\tau = L^{-d}\operatorname{tr}(f(H_\tau)\,\chi_L) = L^{-d} \sum_{v\in V_\tau\cap C_L} (\delta_v, f(H_\tau)\,\delta_v)$$

$$= L^{-d}\int_{C_L} \bar{f}(T_x\tau)\,dx + O(L^{-1})$$

where the last equality uses that the number of vertices within distance $\varepsilon$ of the boundary of $C_L$ is of order $L^{-1}$. Moreover, the $O(L^{-1})$ term can be bounded uniformly in $\tau\in\Omega$ since $\|u-v\|>\varepsilon$ for all $u, v\in V_\tau$ and all $\tau\in\Omega$. Hence (i) follows from (2.1) if we can show that $\bar{f}$ is continuous on $\Omega$.

There exists a constant $K$ such that $\|H_\tau\|<K-1$ for all $\tau\in\Omega$. Then for all $\tau\in\Omega$, the spectrum of $H_\tau$ is strictly contained in $[-K, K]$. For $\eta>0$, let $p$ be a polynomial such that $|f(z)-p(z)|\leqslant\eta$ for all $z\in[-K, K]$. Then $\|f(H_\tau)-p(H_\tau)\|\leqslant\eta$ in the operator norm on $l^2(V_\tau)$ for all $\tau\in\Omega$. Since $(\delta_v, H_\tau\delta_v)$ depends on $\tau$ only through the environment $E_r(v)$ of $v$ up to radius $r$, the quantity $(\delta, p(H_\tau)\,\delta_v)$ depends only on $E_{mr}(v)$, if $p$ is of degree $m$. If $d(\tau, \tau')\leqslant 1/R$, then $\tau$ and $\tau'$ coincide on $B_R$ apart from a translation over $x$ with $\|x\|\leqslant 1/R$. So $V_\tau\cap B_R$ and $V_{\tau'}\cap B_R$ are the same vertexpattern, up to translation, and we can identify their $l^2$-spaces. Hence $(\delta_v, p(H_\tau)\,\delta_v) = (\delta_v, p(H_{\tau'})\,\delta_v)$ if $d(\tau, \tau')\leqslant 1/R$ with $R>mr$ and $v\in B_\varepsilon$. Since $\psi$ is continuous on $\mathbb{R}^d$ it follows that $\bar{f}$ is continuous on $\Omega$. ∎

The theorem can be generalized. The proof applies verbatim to the "vertex-pattern invariant operators" introduced in ref. 1. Also, one could drop the finite-range condition in the definition of vertexpattern invariant operators. It is sufficient to require that $(H_\tau)_\Lambda$ is strongly continuous in $\tau\in\Omega$ on $l^2(\Lambda)$ for every vertexpattern $\Lambda$.

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